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MATHEMATICS

Two Curious Properties on Pairs of M-Gonal Numbers

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Abstract

We search for non-zero positive distinct integers $n_i, i = 1,2,3,...$ such that, the difference between the two polygonal numbers, each having n_i as the rank, is expressed as the difference between two perfect squares. And also, the difference between any two polygonal numbers with the same rank n_i added with the quadratic expression $k^2 + 4k$ is a perfect square.

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Introduction

Number is the essence of mathematical calculation. Variety of numbers has variety of range and richness. Many numbers exhibit fascinating properties, they form sequences, they form patterns and so on [1,2,3]. In [4] explicit formulas for the ranks of Triangular numbers which simultaneously equal to Pentagonal, Octagonal, Decagonal and Dodecagonal numbers in turn are presented. In [5,6,7,8] the authors have obtained the ranks of m-gonal numbers such that the difference between any two m-gonal numbers is unity.

In this communication, we consider two special polygonal numbers with the same rank such that

(i) The difference between the two polygonal numbers added with a square is a perfect square.

(ii) The difference between the two special polygonal numbers added with the quadratic expression $k^2 + 4k$ is a perfect square.

Method of Analysis

Let n be any non-zero positive integer. Let $P_1 (= t_{m,n})$ and $P_2 (= t_{m-4,n})$ be two special polygonal numbers with the same rank n.

Case 1:

Let $P_1 - P_2 = \alpha^2 - k^2; \alpha, k \neq 0$ (1)

$$\Rightarrow 2n^2 - 2n + k^2 = \alpha^2$$

On completing the squares, the above equation is written as $Y^2 = 2\alpha^2 + (1 - 2k^2)$ (2)

where $Y = 2n - 1$ (3)

The initial solution of (2) is $\alpha_0 = k, Y_0 = 1$

To find the other solutions of (2), consider the Pellian equation $Y^2 = 2\alpha^2 + 1$

whose general solution $(\tilde{\alpha}_s, \tilde{Y}_s)$ is given by

$$\tilde{Y}_s = \frac{f_s}{2}; \tilde{\alpha}_s = \frac{g_s}{2\sqrt{2}}$$

where
$$f_s = (3+2\sqrt{2})^{s+1} + (3-2\sqrt{2})^{s+1}$$
$$g_s = (3+2\sqrt{2})^{s+1} - (3-2\sqrt{2})^{s+1}$$

Applying Brahmagupta Lemma between α_0 and Y_0 & $\tilde{\alpha}_s$ and $\tilde{Y}_s$, we have

$$\alpha_{s+1} = \frac{\sqrt{2}kf_s + g_s}{2\sqrt{2}} \quad : \quad Y_{s+1} = \frac{\sqrt{2}f_s + 2kg_s}{2\sqrt{2}}$$

In view of (3), we have

$$n_{s+1} = \frac{Y_{s+1}+1}{2}$$

$$\Rightarrow n_{s+1} = \frac{\sqrt{2}f_s + 2kg_s + 2\sqrt{2}}{4\sqrt{2}} \tag{4}$$

Thus, (4) represents the required values of n such that

$$t_{m,n_{s+1}} - t_{m-4,n_{s+1}} = \alpha_{s+1}^2 - k^2$$

A few illustrations are given in table 1 below

Table:1

m	s	n_{s+1}	$t_{m,n_{s+1}}$	$t_{m-4,n_{s+1}}$	$t_{m,n_{s+1}} - t_{m-4,n_{s+1}} + k^2$
7	0	2(k+1)	$10k^2+17k+7$	$2k^2+5k+3$	$(3k+2)^2$
8	0	2(k+1)	$12k^2+20k+8$	$4k^2+8k+4$	$(3k+2)^2$
7	1	3(4k+3)	$360k^2+522k+189$	$72k^2+114k+45$	$(17k+12)^2$
8	1	3(4k+3)	$432k^2+624k+225$	$144k^2+216k+81$	$(17k+12)^2$

Case 2:

Let $P_1 - P_2 + k^2 + 4k = \alpha^2; \alpha, k \neq 0$ (5)

$$\Rightarrow 2n^2 - 2n + k^2 + 4k = \alpha^2$$

On completing the squares, the above equation is written as

$$Y^2 = 2\alpha^2 + (1 - 2k^2 - 8k) \tag{6}$$

The initial solution of (6) is $\alpha_0 = k+2, Y_0 = 3$

Following the procedure as in case 1, the general solution (α_{s+1}, Y_{s+1}) of (6) is given by

$$\alpha_{s+1} = \frac{\sqrt{2}(k+2)f_s + 3g_s}{2\sqrt{2}} \quad : \quad Y_{s+1} = \frac{3\sqrt{2}f_s + 2(k+2)g_s}{2\sqrt{2}}$$

In view of (3), we have

$$n_{s+1} = \frac{3\sqrt{2}f_s + 2(k+2)g_s + 2\sqrt{2}}{4\sqrt{2}} \tag{4}$$

Thus, (7) represents the required values of n such that

$$t_{m,n_{s+1}} - t_{m-4,n_{s+1}} + k^2 + 4k = \alpha_{s+1}^2$$

A few illustrations are given in table 2 below

Table:2

m	s	n_{s+1}	$t_{m,n_{s+1}}$	$t_{m-4,n_{s+1}}$	$t_{m,n_{s+1}} - t_{m-4,n_{s+1}} + (k^2+4k)$
7	0	2k+9	$10k^2+87k+189$	$2k^2+19k+45$	$(3k+12)^2$
8	0	2k+9	$12k^2+104k+225$	$4k^2+36k+81$	$(3k+12)^2$
7	1	12k+50	$360k^2+2982k+6175$	$72k^2+606k+1275$	$(17k+70)^2$
8	1	12k+50	$432k^2+3576k+7400$	$144k^2+1200k+2500$	$(17k+70)^2$

Remark:

For simplicity and clear understanding we present in table III below, curious properties on some more pairs of m-gonal number.

Table III: Pairs of polygonal numbers with properties

Pairs of polygonal numbers	Observations	n	$\{f_s, g_s\}$
$(t_{m,n}, t_{m-1,n})$, $m \geq 4$	$t_{m,n} - t_{m-1,n} + k^2 = \alpha^2$	$\frac{1}{4}\left[f_s + 2\sqrt{2}kg_s + 2\right]$; $s = 0,1,2,\ldots\ldots.$	$\left\{\begin{matrix}(3+\sqrt{8})^{s+1} + (3-\sqrt{8})^{s+1}, \\ (3+\sqrt{8})^{s+1} - (3-\sqrt{8})^{s+1}\end{matrix}\right\}$
	$t_{m,n} - t_{m-1,n} + k^2 + 2k = \alpha^2$	$\frac{1}{4}\left[3f_s + 2\sqrt{2}(k+1)g_s + 2\right]$; $s = 0,1,2,\ldots\ldots.$	
$(t_{m,n}, t_{m-3,n})$, $m \geq 6$	$t_{m,n} - t_{m-3,n} + k^2 = \alpha^2$	$\frac{1}{12}\left[3f_s + 2\sqrt{6}kg_s + 6\right]$; $s = 0,1,2,\ldots\ldots.$	$\left\{\begin{matrix}(5+\sqrt{24})^{s+1} + (5-\sqrt{24})^{s+1}, \\ (5+\sqrt{24})^{s+1} - (5-\sqrt{24})^{s+1}\end{matrix}\right\}$
	$t_{m,n} - t_{m-3,n} + k^2 + 6k = \alpha^2$	$\frac{1}{12}\left[15f_s + 2\sqrt{6}(k+3)g_s + 6\right]$; $s = 0,1,2,\ldots\ldots.$	
$(t_{m,n}, t_{m-5,n})$, $m \geq 8$	$t_{m,n} - t_{m-5,n} + k^2 = \alpha^2$	$\frac{1}{20}\left[5f_s + 2\sqrt{10}kg_s + 10\right]$; $s = 0,1,2,\ldots\ldots.$	$\left\{\begin{matrix}(19+3\sqrt{40})^{s+1} + (19-3\sqrt{40})^{s+1}, \\ (19+3\sqrt{40})^{s+1} - (19-3\sqrt{40})^{s+1}\end{matrix}\right\}$
	$t_{m,n} - t_{m-5,n} + k^2 + 30k = \alpha^2$	$\frac{1}{20}\left[95f_s + 2\sqrt{10}(k+15)g_s + 10\right]$; $s = 0,1,2,\ldots\ldots.$	
$(t_{m,n}, t_{m-7,n})$, $m \geq 10$	$t_{m,n} - t_{m-7,n} + k^2 = \alpha^2$	$\frac{1}{28}\left[7f_s + 2\sqrt{14}kg_s + 14\right]$; $s = 1,3,5,\ldots\ldots.$	$\left\{\begin{matrix}(15+2\sqrt{56})^{s+1} + (15-2\sqrt{56})^{s+1}, \\ (15+2\sqrt{56})^{s+1} - (15-2\sqrt{56})^{s+1}\end{matrix}\right\}$
$(t_{m,n}, t_{m-8,n})$, $m \geq 11$	$t_{m,n} - t_{m-8,n} = (4a^2-1)k^2$	$\frac{1}{4}\left[f_s + 2\right]$; $s = 1,3,5,\ldots\ldots.$	$\left\{\begin{matrix}\left(2a+\sqrt{4a^2-1}\right)^{s+1} + \left(2a-\sqrt{4a^2-1}\right)^{s+1}, \\ \left(2a+\sqrt{4a^2-1}\right)^{s+1} - \left(2a-\sqrt{4a^2-1}\right)^{s+1}\end{matrix}\right\}$
	$t_{m,n} - t_{m-8,n} = (4a^2+4a)k^2$	$\frac{1}{4}\left[f_s + 2\right]$; $s = 0,2,4,6,\ldots\ldots.$	$\left\{\begin{matrix}\left(2a+\sqrt{4a^2-1}\right)^{s+1} + \left(2a-\sqrt{4a^2-1}\right)^{s+1}, \\ \left(2a+\sqrt{4a^2-1}\right)^{s+1} - \left(2a-\sqrt{4a^2-1}\right)^{s+1}\end{matrix}\right\}$

	$t_{m,n} - t_{m-8,n} + k^2 - 2k + 2 = \alpha^2$; $k = 8cd + 4d + 1$, $\alpha = (2c+1)^2 + 4d^2, c > d > 0$	$2c^2 + 2c - 2d^2 + 1$	-
	$t_{m,n} - t_{m-8,n} + k^2 = \alpha^2$ (i). $k = 3k + 2, \alpha = 5k + 2$ (ii). $k = t_{3,n-1} - 2, \alpha = t_{3,n-1} + 2$	(i). $2k+1$ (ii). $n > 1$	-
$(t_{m,n}, t_{m-9,n})$, $m \geq 12$	$t_{m,n} - t_{m-9,n} + k^2 = \alpha^2$	$\frac{1}{6}\left[\frac{3}{2}f_s + k\sqrt{2}g_s + 3\right]$; $s = 1,3,5,\ldots\ldots$	$\left\{\begin{matrix}(3+\sqrt{8})^{s+1} + (3-\sqrt{8})^{s+1}, \\ (3+\sqrt{8})^{s+1} - (3-\sqrt{8})^{s+1}\end{matrix}\right\}$

Conclusion

In this paper, we have presented pairs of special polygonal numbers each with the same rank satisfying specific relations. As polygonal numbers are rich in variety, one may attempt for finding other choices for pairs of polygonal numbers with different relations. The above analysis may also be considered for higher ordered polygonal numbers.

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