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MATHEMATICS

OBSERVATION ON THE BIQUADRATIC EQUATION WITH FIVE UNKNOWNS

$$4x^3+4y^3-2x^2y-2xy^2=23p^2(z^2-w^2)$$

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Abstract

We present different patterns of non-zero distinct integer solutions to the non- homogeneous biquadratic equation with five unknowns given by $4x^3+4y^3-2x^2y-2xy^2=23p^2(z^2-w^2)$ *A few interesting properties among the solutions are also given*.

Keywords: Biquadratic, Biquadratic with five unknowns, integer solutions.

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Notation:

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$
$$p_n^m=\frac{n(n+1)}{6}[n(m-2)+5-m)]$$
$$pr_n=n(n+1)$$
$$S_n=6n(n-1)+1$$
$$SO_n=n(2n^2-1)$$
$$j_n=2^n+(-1)^n$$
$$J_n=\frac{1}{3}[2^n+(-1)^n]$$

Introduction:

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, biquadratic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context one may refer [6-9] for various problems on the biquadratic Diophantine equation with four variables and [10-14] for five variables and [15-16] for six variables. It is towards this end, this paper concerns with the problem of determining non-trival integral solutions of the homogeneous equation of degree four with six unknowns given by $4x^3+4y^3-2x^2y-2xy^2=23p^2(z^2-w^2)$.

Method of Analysis:

The biquadratic equation with five unknowns to be solved is

$$4x^3+4y^3-2x^2y-2xy^2=23p^2(z^2-w^2) \tag{1}$$

To start with, it is observed that (1) is satisfied by the following quintuples

$$(x,y,z,w,p)=(424T,216T,320+1,320T-1,88T),(70T,-6T,32T+1,32T-1,22T),$$
$$(10K^2+56K-378,6K^2-40K-262,8K^2+8K-321,8K^2+8K-319,2K^2+16K-74),$$
$$(1610K^2+1656K+324,966K^2+920K-216,1288K^2+1288K+321,1288K^2+1288K+321,$$
$$322K^2+336K+88),(230K^2+276K+17,138K^2+92K+47,184K^2+184K+33,184K^2+184K+31,$$
$$46K^2+60K+22)$$

However, we have other choices of integer solution to (1) which are illustrated as follows:
Introducing, the linear transformation

$$x=u+v,\ y=u-v,\ z=u+1,\ w=u-1 \quad (2)$$

in (1), it is written as

$$u^2+7v^2=23p^2 \quad (3)$$

(3) is solved through different methods leading to different sets of integer solutions to (1)

Method 1:

Assume $p=a^2+7b^2$ (4)

Write 23 as $23=(4+i\sqrt{7})(4-i\sqrt{7})$ (5)

Using (4) & (5) in (3) and applying the method of factorization, define

$$u+i\sqrt{7}v=(4+i\sqrt{7})(a+i\sqrt{7}b)^2$$

Equating the real and imaginary parts, we have

$$u=4a^2-35b^2-6ab$$
$$v=a^2-7b^2+8ab$$

Substituting the above values of u and v in (2), the values of x,y,z,w and p are given by

$$\left.\begin{aligned} x&=x(a,b)=5a^2-35b^2-6ab\\ y&=y(a,b)=3a^2-21b^2-22ab\\ z&=z(a,b)=4a^2-28b^2-14ab+1\\ w&=w(a,b)=4a^2-28b^2+14ab-1 \end{aligned}\right\} \quad (6)$$

Thus (4) and (6) represent the required integer solutions to (1).

Properties:

1. $x(t_{3,a+1},1)-y(t_{3,a+1},1)-p(t_{3,a+1},1)+21={t_{3,a+1}}^2+48\,pr_a^3$
2. $3p(a,a^2)$ is a Nasty Number.
3. $p(a,a+1)+y(a,a+1)-z(a,a+1)-s_a+16t_{3,a}-2t_{10,a}\equiv -28 \pmod{40}$

Method 2:
Rewrite (3) as

$$u^2+7v^2=23p^2*1 \quad (7)$$

Write 1 as $1=\dfrac{(3+i\sqrt{7})(3-i\sqrt{7})}{16}$ (8)

Following the procedure presented above in Method 1, the corresponding integer solutions to (1) are given by

$$x = x(A,B) = 48A^2 - 352AB - 336B^2$$
$$y = y(A,B) = -8A^2 - 432AB + 56B^2$$
$$z = z(A,B) = 20A^2 - 392AB - 140B^2 + 1$$
$$w = w(A,B) = 20A^2 - 392AB - 140B^2 - 1$$
$$p = p(A,B) = 16A^2 + 112B^2$$

Properties:

1. $x(a,1) - z(a,1) - w(a,1) - 8pr_a \equiv 368 \pmod{424}$
2. $z(a^2,1) - p(a^2,1) - t_{10,a} \equiv 144 \pmod{395}$
3. $3[y(A,A+1) + p(A,A+1) + 640t_{3,A}]$ is a Nasty Number.
4. $z(a+1,a) - x(a+1,a) - y(a+1,a) - 20S_a - 392pr_a \equiv -79 \pmod{80}$.

Note:

Instead (7), one may write 1 in two different ways as

$$1 = \frac{(1+i3\sqrt{7})(1-i3\sqrt{7})}{64} \quad ; \quad 1 = \frac{(3+i20\sqrt{7})(3-i20\sqrt{7})}{53^2}$$

In this case, corresponding two sets of non zero distinct integer solutions to (1) are found to be

Set 1:

$$x = x(A,B) = -32A^2 - 1728AB + 224B^2$$
$$y = y(A,B) = -240A^2 - 1184AB + 1680B^2$$
$$z = z(A,B) = -136A^2 - 1456AB + 952B^2 + 1$$
$$w = w(A,B) = -136A^2 - 1456AB + 952B^2 - 1$$
$$p = p(A,B) = 64A^2 + 448B^2$$

Set 2:

$$x = x(A,B) = -2385A^2 - 75154AB + 16695B^2$$
$$y = y(A,B) = -11183A^2 - 48018AB + 78281B^2$$
$$z = z(A,B) = -6784A^2 - 61586AB + 47488B^2 + 1$$
$$w = w(A,B) = -6784A^2 - 61586AB + 47488B^2 - 1$$
$$p = p(A,B) = 2809A^2 + 19663B^2$$

Method 3:

Introducing, the linear transformation

$$u = 4U, p = X + 7T, v = X + 23T \qquad (9)$$

in (3), it leads to

$$X^2 = U^2 + 161T^2 \qquad (10)$$

Which is satisfied by

$$T = 2rs$$
$$U = 161r^2 - s^2$$
$$X = 161r^2 + s^2$$

Using the above U, T, X in (9) and (2) , the corresponding non zero integer solutions to (1) are given by

$$x = x(r,s) = 805r^2 + 46rs - 3s^2$$

$$y = y(r,s) = 483r^2 - 46rs - 5s^2$$

$$z = z(r,s) = 644r^2 - 4s^2 + 1$$

$$w = w(r,s) = 644r^2 - 4s^2 - 1$$

$$p = p(r,s) = 161r^2 + 14rs + s^2$$

Properties:

1. $x(\alpha, t_{3,\alpha+1}) - w(\alpha, t_{3,\alpha+1}) - p(\alpha, t_{3,\alpha+1}) = 96p_{\alpha}^3 + 1$
2. $x(2^n,1) - y(2^n,1) - p(2^n,1) = 47J_{2n} + 117J_n + 162$

Note:

In (9), consider p and v as $p = X - 7T, v = X - 23T.$

For this choice, the corresponding integer solutions to (1) are given by

$$x = x(r,s) = 805r^2 - 46rs - 5s^2$$

$$y = y(r,s) = 483r^2 + 46rs - 3s^2$$

$$z = z(r,s) = 644r^2 - 4s^2 + 1$$

$$w = w(r,s) = 644r^2 - 4s^2 - 1$$

$$p = p(r,s) = 161r^2 - 14rs - s^2$$

Conclusion:

In this paper, we have illustrated different methods of obtaining integer solutions to the b biquadratic equation with five unknowns are rich in variety, one may search for the non zero distinct integer solutions for other choices of biquadratic equation with five or more unknowns.

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