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A CONNECTION BETWEEN TRIANGULAR NUMBER AND FRUSTUM OF THE CONE

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Abstract

This paper concerns with the problem of obtaining the radii of the bottom and top circles of the frustum of the cone of given height 'h' in connection with the triangular number and the special characterization of the frustum of the considered cone.

Keywords: Triangular number, Frustum of a cone, Pythagorean equation.

SUBJECT CLASSIFICATION:

2010 Mathematics Subject Classification:11D09.

INTRODUCTION:

Diophantine equations are very often polynomials with integer coefficients in which the solutions are required to be integers. They have been studied since antiquity and are mathematically both challenging and attractive because of the great diversity of methods that are needed to understand them [1-9].A careful observe of patterns may note that there is a one to one correspondence between the polygonal numbers and the number of sides of the polygon. In [10-12], Special Pythagorean triangles connected with polygonal numbers are obtained. In [13] Special Pythagorean triangles in connection with Hardy Ramanujan number 1729 are exhibited. Pythagorean triangles in connections with 5 digit Dhuruva numbers namely 53955,59994 are given. In [14], The authors have studied the connection between rectangle and Dhuruva numbers of digits 3 and 5. In [15], Rectangle with area as a special polygonal number has been analyzed. This paper concerns with the problem of obtaining the radii of the bottom and top circles of the frustrum of the cone of given height 'h' in connection with the triangular number and the special characterization of the frustum of the considered cone.

NOTATIONS:

Volume of the frustum of a cone, V= $\frac{\pi h}{3}[R^2 + r^2 + Rr]$

Lateral surface area of frustum of the cone, LS= $\pi l(R+r)$

Total surface area of frustum of the cone, TS= $\pi[R^2 + r^2 + Rr]$

h= Height of the frustum of the cone
r = Radius of the upper circle
R= Radius of the base circle

$$t_{3,r} = \frac{r(r+1)}{2}$$

METHOD OF ANALYSIS:

SECTION A:

Consider the frustum of the cone with given height 'h'. Let R, r be respectively be the radii of the bottom and top circles of the frustum to be determined such that

$$\frac{3}{\pi}\left(\frac{V}{h} + TS - LS\right) - Rr - 4 = 16t_{3,r} + \alpha^2, \quad \alpha > 0 \tag{1}$$

Substituting the values of V, T.S and L.S in (1), it is written as

$$(2R)^2 = (2r+2)^2 + \alpha^2 \tag{2}$$

The above equation is in the form of the well- known Pythagorean equation which is satisfied by

$$r = uv - 1$$
$$\alpha = u^2 - v^2$$
$$R = \frac{1}{2}(u^2 + v^2) \quad , u > v > 0$$

As the values of R, r should be integers , it is noted that the values of u and v should be of the same parity.

CASE 1:

Let $u = 2U$, $v = 2V$. The corresponding values of α , R, r are given by

$$\alpha = 4(U^2 - V^2)$$
$$R = 2(U^2 + V^2) \quad (3)$$
$$r = 4UV - 1 \quad (4)$$

Thus the radii of the bottom and top circles of the frustum satisfying (1) are given by (3) and (4).

For clear understanding, a few numerical illustration are presented in table-1 below in which $X = \frac{3}{\pi}(\frac{V}{h} + TS - LS)$, $Y = Rr + 4$.

Table:1

U	V	α	R	R	X	Y	$X - Y$
2	1	12	10	7	666	74	592=$16t_{3,7} + (12)^2$
4	3	28	50	47	21186	2354	18832=$16t_{3,47} + (28)^2$
6	5	44	122	119	130698	14518	116176=$16t_{3,119} + (44)^2$
8	7	60	226	223	453618	50402	403216=$16t_{3,223} + (60)^2$

CASE 2:

Let $u = 2U + 1$, $v = 2V + 1$. The corresponding values of α ,R ,r are given by

$$\alpha = 4U^2 - 4V^2 + 4U - 4V$$
$$R = 2U^2 + 2V^2 + 2U + 2V + 1 \quad (5)$$
$$r = 4UV + 2U + 2V \quad (6)$$

Thus the radii of the bottom and top circles of the frustum satisfying (1) are given by (5) and (6).

For clear understanding, a few numerical illustrations are presented in table-2 below.

TABLE :2

U	V	α	R	r	X	Y	$X - Y$
2	1	16	17	14	2178	232	1936=$16t_{3,14} + (16)^2$
4	3	32	65	62	36306	4034	32272=$16t_{3,62} + (32)^2$
6	5	48	145	142	185346	20594	164752=$16t_{3,142} + (48)^2$
8	7	64	257	254	587538	65282	522256=$16t_{3,254} + (64)^2$

NOTE:

Alternatively, one may solve (2) using fibonacci numbers and the process is as follows:
Let p,q,t,s be four consecutive fibonacci numbers so that

$$ps = t^2 - q^2$$

Squaring both sides left side and performing some simple algebra, we have

$$(t^2 + q^2) = (ps)^2 + (2qt)^2$$

Comparing this equation with (2), we get

$$\alpha = ps$$

$$R = \frac{1}{2}(t^2 + q^2) \qquad (7)$$

$$r = qt - 1 \qquad (8)$$

It is worth to note that, when p is even the values of 'R' are integers for q even or odd. Considering the corresponding values of q, t in (7) and (8) , the required values of R, r satisfying (1) are obtained.

METHOD 2:

Consider the frustum of the cone with given height 'h'. Let R, r be respectively be the radii of the bottom and top circles of the frustum to be determined such that

$$\frac{3}{\pi}(\frac{V}{h} + TS - LS) - Rr - 4 = 16t_{3,r} + 7\alpha^2, \ \alpha > 0 \qquad (9)$$

Substituting the values of V, TS and LS in (1), it is written as

$$4R^2 = (2r+2)^2 + 7\alpha^2 \qquad (10)$$

In what follows, we present different ways of solving (10), leading to different sets of values for R, r satisfying (9).

SET 1:

Take $R = a^2 + 7b^2$ (11)

Write '4' as

$$4 = \frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{4} \qquad (12)$$

Substituting (11) and (12) in (10) and using the method of factorization, define

$$(2r+2) + i\sqrt{7}\alpha = \frac{(3 + i\sqrt{7}\alpha)}{2}(a + i\sqrt{7}b)^2 \qquad (13)$$

Equating real and imaginary parts of (13),we get

$$r = \frac{3}{4}a^2 - \frac{21}{4}b^2 - \frac{7}{2}ab - 1 \qquad (14)$$

and

$$\alpha = 3ab + \frac{1}{2}a^2 - \frac{7}{2}b^2 \qquad (15)$$

Replacing 'a' by 2A and 'b' by 2B in (14),(15) and (11), we get

$$R = 4A^2 + 28B^2$$

$$r = 3A^2 - 21B^2 - 14AB - 1$$

$$\alpha = 12AB + 2A^2 - 14B^2$$

Note that the above values of R, r satisfy (9). A few numerical examples are given in table-3 below.

TABLE-3:

A	B	α	R	R	X	Y	$X-Y$
6	1	130	172	2	118696	348	118348=$16t_{3,2}+7(130)^2$
12	2	520	688	11	19101428	7572	1893856=$16t_{3,11}+7(520)^2$
18	3	1170	1548	26	9628168	40252	9587916=$16t_{3,26}+7(1170)^2$
24	4130	2080	2752	47	30432196	129348	30302848=$16t_{3,47}+7(2080)^2$

NOTE:

It is worth to mention here that in (13), 4 may also be represented as follows:

$$4=\frac{(1+i3\sqrt{7})(1-i3\sqrt{7})}{16}$$

Following the procedure presented above, the corresponding values of R, r are given by

$$R=16A^2+112B^2$$

$$r=2A^2-14B^2-84AB-1$$

$$\alpha=8AB+12A^2-84B^2$$

SET 2:

Note that (10) is in the form $z^2=Dx^2+y^2, D>0$ and square-free.

Thus, the solutions of (10) are given by

$$\alpha=2uv$$

$$r=\frac{1}{2}(7u^2-v^2-2)$$

$$R=\frac{7u^2+v^2}{2},\ u>v>0$$

As the values of R, r should be integers , it is noted that the values of u and v should be of the same parity.

CASE 1:

Let $u=2U$, $v=2V$. The corresponding values of α, R, r are given by

$$\alpha=8UV$$

$$R=2(7U^2+V^2) \quad (16)$$

$$r=14U^2-2V^2-1 \quad (17)$$

Thus the radii of the bottom and top circles of the frustum satisfying (9) are given by (16) and (17).

For clear understanding, a few numerical illustrations are presented in table-4 below.

TABLE-4:

U	V	α	R	r	X	Y	$X-Y$
1	2	16	22	5	2146	114	2032=$16t_{3,5}+7(16)^2$
2	4	96	158	93	149146	14698	134448=$16t_{3,93}+7(96)^2$
3	6	240	422	277	1136146	116898	1019248=$16t_{3,277}+7(240)^2$
4	8	448	814	557	4344778	453402	3891376=$16t_{3,557}+7(448)^2$

CASE 2:

Let $u = 2U + 1$ and $v = 2V + 1$. The corresponding values of α , R, r are given by

$$\alpha = 8UV + 4U + 4V + 2$$

$$R = 14U^2 + 14U + 2V^2 + 2V + 4 \quad (18)$$

$$r = 14U^2 + 14U - 2V^2 - 2V + 2 \quad (19)$$

Thus the radii of the bottom and top circles of the frustum satisfying (9) are given by (18) and (19).

For clear understanding, a few numerical illustrations are presented in table-5 below.

TABLE :5

U	V	α	R	R	X	Y	$X - Y$
1	2	30	44	18	9832	796	9036=$16t_{3,18} + 7(30)^2$
3	4	126	212	130	274936	27564	247372=$16t_{3,130} + 7(126)^2$
5	6	286	508	338	1660936	171708	1489228=$16t_{3,338} + 7(286)^2$
7	8	510	932	642	5721496	598348	5123148=$16t_{3,642} + 7(510)^2$

CONCLUSION:

In this paper, we have exhibited connection between triangular number and special characterization of the frustum of the cone. As there are many 3D geometrical representations, one may consider connections between special polygonal numbers with other geometrical figures.

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