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MATHEMATICS

ON THE DIOPHANTINE EQUATION y^2=$3x^2$+6

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Abstract

The binary quadratic equation represented by the positive pellian $y^2=3x^2+6$ *is analysed for its distinct integer solutions. A few interesting relations among the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.*

Keywords: Binary quadratic, hyperbola, parabola, integral solutions, pell equation.
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INTRODUCTION

The binary quadratic equation of the form $y^2=Dx^2+1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2=3x^2+6$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

METHOD OF ANALYSIS

The positive pell equation representing hyperbola under consideration is,

$$y^2=3x^2+6 \tag{1}$$

The smallest positive integer solution of (1) is

$x_0=1,\ y_0=3$

To obtain the other solution of (1), consider the pellian equation

$y^2=3x^2+1$

whose general solution $(\widetilde{x_n}\ ,\widetilde{y_n})$ is given by

$$\widetilde{x_n}=\frac{1}{2\sqrt{3}}g_n \text{ and } \widetilde{y_n}=\frac{1}{2}f_n \tag{2}$$

where,

$f_n=\left(2+\sqrt{3}\right)^{n+1}+\left(2-\sqrt{3}\right)^{n+1}$

$g_n=\left(2+\sqrt{3}\right)^{n+1}-\left(2-\sqrt{3}\right)^{n+1}$, n=0,1,2,3…..

Applying Brahamagupta lemma between (x_0,y_0) and $(\tilde{x}_n,\tilde{y}_n)$, the other integer solutions of (1) are given by $x_{n+1}=\frac{1}{2}f_n+\frac{\sqrt{3}}{2}g_n$; $y_{n+1}=\frac{3}{2}f_n+\frac{\sqrt{3}}{2}g_n$

The recurrence relations satisfied by the solutions (2) are given by

$$x_{n+3}-4x_{n+2}+x_{n+1}=0$$
$$y_{n+3}-4y_{n+2}+y_{n+1}=0$$

Some numerical examples of x and y satisfying (1)are given in the Table 1 below:

TABLE 1: Examples

n	x_n	y_n
0	1	3

1	5	9
2	19	33
3	71	123
4	265	459
5	989	1713
6	3691	6393

From the above table, we observe some interesting relations among the solutions which are presented below:

1) Both x_n and y_n are odd.
2) Each of the following expression is a nasty number:
   - $6(x_{2n+3} - 3x_{2n+2} + 2)$
   - $\frac{3}{2}(x_{2n+4} - 11x_{2n+2} + 8)$
   - $3(y_{2n+3} - 5x_{2n+2} + 4)$
   - $\frac{6}{7}(y_{2n+4} - 19x_{2n+2} + 14)$
   - $6(3x_{2n+4} - 11x_{2n+3} + 2)$
   - $3(3y_{2n+2} - x_{2n+3} + 4)$
   - $6(3y_{2n+3} - 5x_{2n+3} + 2)$
   - $3(3y_{2n+4} - 19x_{2n+3} + 4)$
   - $\frac{6}{7}(11y_{2n+2} - x_{2n+4} + 14)$
   - $3(11y_{2n+3} - 5x_{2n+4} + 4)$
   - $6(11y_{2n+4} - 19x_{2n+4} + 2)$
   - $2(5y_{2n+2} - y_{2n+3} + 6)$
   - $\frac{1}{2}(19y_{2n+2} - y_{2n+4} + 24)$
   - $2(19y_{2n+3} - 5y_{2n+4} + 6)$

3) Each of the following expressions is a cubical integer:
   - $(x_{3n+4} - 3x_{3n+3}) + 3(x_{n+2} - 3x_{n+1})$
   - $16\{(x_{3n+5} - 11x_{3n+3}) + 3(x_{n+3} - 11x_{n+1})\}$
   - $4\{(y_{3n+4} - 5x_{3n+3}) + 3(y_{n+2} - 5x_{n+1})\}$
   - $49\{(y_{3n+5} - 19x_{3n+3}) + 3(y_{n+3} - 19x_{n+1})\}$
   - $(3x_{3n+5} - 11x_{3n+4}) + 3(3x_{n+3} - 11x_{n+2})$
   - $4\{(3y_{3n+3} - x_{3n+4}) + 3(3y_{n+1} - x_{n+2})\}$
   - $(3y_{3n+4} - 5x_{3n+4}) + 3(3y_{n+2} - 5x_{n+2})$
   - $4\{(3y_{3n+5} - 19x_{3n+4}) + 3(3y_{n+3} - 19x_{n+2})\}$
   - $49\{(11y_{3n+3} - x_{3n+5}) + 3(11y_{n+1} - x_{n+3})\}$
   - $4\{(11y_{3n+4} - 5x_{3n+5}) + 3(11y_{n+2} - 5x_{n+3})\}$
   - $(11y_{3n+5} - 19x_{3n+5}) + 3(11y_{n+3} - 19x_{n+3})$
   - $9\{(5y_{3n+3} - y_{3n+4}) + 3(5y_{n+1} - y_{n+2})\}$
   - $144\{(19y_{3n+3} - y_{3n+5}) + 3(19y_{n+1} - y_{n+3})\}$
   - $9\{(19y_{3n+4} - 5y_{3n+5}) + 3(19y_{n+2} - 5y_{n+3})\}$

4) Each of the following represents the relations among the solutions:
   - $x_{n+3} = 4x_{n+2} - x_{n+1}$
   - $y_{n+1} = x_{n+2} - 2x_{n+1}$
   - $y_{n+2} = 2x_{n+2} - x_{n+1}$
   - $y_{n+3} = 7x_{n+2} - 2x_{n+1}$
   - $4y_{n+1} = x_{n+3} - 7x_{n+1}$
   - $2y_{n+2} = x_{n+3} - x_{n+1}$
   - $4y_{n+3} = 7x_{n+3} - x_{n+1}$
   - $2y_{n+1} = y_{n+2} - 3x_{n+1}$
   - $2y_{n+3} = 7y_{n+2} + 3x_{n+1}$
   - $7y_{n+1} = y_{n+3} - 12x_{n+1}$

- $y_{n+1} = 2x_{n+3} - 7x_{n+2}$
- $y_{n+2} = x_{n+3} - 2x_{n+2}$
- $y_{n+3} = 2x_{n+3} - x_{n+2}$
- $2y_{n+2} = y_{n+1} + 3x_{n+2}$
- $y_{n+3} = y_{n+1} + 6x_{n+2}$
- $y_{n+3} = 2y_{n+2} + 3x_{n+2}$
- $7y_{n+2} = 2y_{n+1} + 3x_{n+3}$
- $7y_{n+3} = y_{n+1} + 12x_{n+3}$
- $2y_{n+3} = y_{n+2} + 3x_{n+3}$
- $y_{n+3} = 4y_{n+2} - y_{n+1}$

REMARKABLE OBSERVATION

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table 2 below

TABLE 2: HYPERBOLA

S.NO	(X,Y)	HYPERBOLA
1	$(5x_{n+1} - x_{n+2}, x_{n+2} - 3x_{n+1})$	$3Y^2 - X^2 = 12$
2	$(19x_{n+1} - x_{n+3}, x_{n+3} - 11x_{n+1})$	$3Y^2 - X^2 = 192$
3	$(9x_{n+1} - y_{n+2}, y_{n+2} - 5x_{n+1})$	$3Y^2 - X^2 = 48$
4	$(33x_{n+1} - y_{n+3}, y_{n+3} - 19x_{n+1})$	$3Y^2 - X^2 = 588$
5	$(19x_{n+2} - 5x_{n+3}, 3x_{n+3} - 11x_{n+2})$	$3Y^2 - X^2 = 12$
6	$(3x_{n+2} - 5y_{n+1}, 3y_{n+1} - x_{n+2})$	$3Y^2 - X^2 = 48$
7	$(18x_{n+2} - 10y_{n+2}, 3y_{n+2} - 5x_{n+2})$	$3Y^2 - X^2 = 12$
8	$(33x_{n+2} - 5y_{n+3}, 3y_{n+3} - 19x_{n+2})$	$3Y^2 - X^2 = 48$
9	$(3x_{n+3} - 19y_{n+1}, 11y_{n+1} - x_{n+3})$	$3Y^2 - X^2 = 588$
10	$(9x_{n+3} - 19y_{n+2}, 11y_{n+2} - 5x_{n+3})$	$3Y^2 - X^2 = 48$
11	$(33x_{n+3} - 19y_{n+3}, 11y_{n+3} - 19x_{n+3})$	$3Y^2 - X^2 = 12$
12	$(y_{n+2} - 3y_{n+1}, 5y_{n+1} - y_{n+2})$	$Y^2 - 3X^2 = 36$
13	$(19y_{n+1} - y_{n+3}, y_{n+3} - 11y_{n+1})$	$Y^2 - 3X^2 = 576$
14	$(3y_{n+3} - 11y_{n+2}, 19y_{n+2} - 2y_{n+3})$	$Y^2 - 3X^2 = 36$

II. Employing linear combinations among the solutions of (1) , one may generate integer solutions for other choices of parabolas which are presented in the Table 3 below

TABLE 3: PARABOLAS

S.NO	(X,Y)	PARABOLA
1	$(5x_{n+1} - x_{n+2}, x_{2n+3} - 3x_{2n+2} + 2)$	$X^2 = 3Y - 12$
2	$(19x_{n+1} - x_{n+3}, x_{2n+4} - 11x_{2n+2} + 8)$	$X^2 = 2Y - 192$
3	$(9x_{n+1} - y_{n+2}, y_{2n+3} - 5x_{2n+2} + 4)$	$X^2 = 6Y - 48$
4	$(33x_{n+1} - y_{n+3}, y_{2n+3} - 19x_{2n+2} + 14)$	$X^2 = 21Y - 288$
5	$(19x_{n+2} - 5x_{n+3}, 3x_{2n+4} - 11x_{2n+3} + 2)$	$X^2 = 3Y - 12$
6	$(3x_{n+2} - 5y_{n+1}, 3y_{2n+2} - x_{2n+3} + 4)$	$X^2 = 6Y - 48$

7	$(18x_{n+2} - 10y_{n+2}\ , 3y_{2n+3} - 5x_{2n+3} + 2)$	$X^2 = 3Y - 12$
8	$(33x_{n+2} - 5y_{n+3}\ , 3y_{2n+4} - 19x_{2n+3} + 4)$	$X^2 = 6Y - 48$
9	$(3x_{n+3} - 19y_{n+1}\ , 11y_{2n+2} - x_{2n+4} + 14)$	$X^2 = 21Y - 588$
10	$(9x_{n+3} - 19y_{n+2}\ , 11y_{2n+3} - 5x_{2n+4} + 4)$	$X^2 = 6Y - 48$
11	$(33x_{n+3} - 19y_{n+3}\ , 11y_{2n+4} - 19x_{2n+4} + 2)$	$X^2 = 3Y - 12$
12	$(y_{n+2} - 3y_{n+1}\ , 5y_{2n+2} - y_{2n+3} + 6)$	$X^2 = Y - 12$
13	$(19y_{n+1} - y_{n+3}\ , 19y_{2n+2} - y_{2n+4} + 24)$	$X^2 = 4Y - 192$
14	$(3y_{n+3} - 11y_{n+2}\ , 19y_{2n+3} - 5y_{2n+4} + 6)$	$X^2 = Y - 12$

III. Consider p= $x_{n+1} + y_{n+1}\ , q = x_{n+1}$, observe that p> q > 0. Treat p,q as the generators of the Pythagorean Triangle T(α, β, γ),
where

$$\alpha = 2pq\ , \beta = p^2 - q^2\ , \gamma = p^2 + q^2$$

Then the following interesting relations are observed

- ❖ $2\alpha - 3\beta + \gamma = -12$
- ❖ $5\alpha - 2\gamma + 12 = \frac{12A}{P}$
- ❖ $\frac{2A}{P} = x_{n+1}y_{n+1}$

CONCLUSION

In this paper, we have presented infinitely many integer solution for the hyperbola represented by the positive pellequation $y^2 = 3x^2 + 6$. As the binary quadratic Diophantine equation are rich in variety, one may search for the other choices of positive pell equation and determine their integer solution along with suitable properties.

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