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MATHEMATICS

ON THE HOMOGENEOUS TERNARY QUADRATIC EQUATION

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ABSTRACT

The ternary quadratic equation given by is considered and searched for its many different integer solution. Five different choices of integer solution of the above equations are presented. A few interesting relations between the solutions and special polygonal numbers are presented.

KEYWORDS: Ternary Quadratic, Integer Solutions

MSC subject classification: 11D09

INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-8] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation representing homogeneous equation with three for determining its infinitely many non-zero integral points. Also, few interesting relations among the solutions are presented.

NOTATIONS:

- $t_{m,n} = n^{th}$ term of a regular polygon with m sides.
  $= n\left(1+\frac{(n-1)(m-2)}{2}\right)$
- P_{r_n} =pronic number of rank n
  $= n(n+1)$

METHOD OF ANALYSIs

The Quadratic Diophantine equation with three unknowns to be solved is given by

$$x^2 + 10xy + 32y^2 = 8z^2 \qquad (1)$$

$$(x^2 + 10xy + 25y^2) + 7y^2 = 8z^2$$

$$(x+5y)^2 + 7y^2 = 8z^2 \quad (2)$$

The substituting

$$U = x + 5y \qquad (3)$$

in (1) we get,

$$U^2 + 7y^2 = 8z^2 \qquad (4)$$

(4) Is solved through different approaches and the different patterns of solutions of (1) obtained are presented below.

PATTERN: 1

Assume

$$z = a^2 + 7b^2 \qquad (5)$$

Write '8' as

$$8 = (1+i\sqrt{7})(1-i\sqrt{7}) \qquad (6)$$

Substituting (5) and (6) in (4), we get

$$(U+i\sqrt{7}y)(U-i\sqrt{7}y) = (1+i\sqrt{7})(1-i\sqrt{7})(a+i\sqrt{7}b)^2(a-i\sqrt{7}b)^2$$

Consider the positive factor

$$U + i\sqrt{7}y = (1+i\sqrt{7})(a+i\sqrt{7}b)^2$$

$$(U+i\sqrt{7}y) = (a^2 - 7b^2 + 2abi\sqrt{7})(1+i\sqrt{7})$$

$$(U+i\sqrt{7}y) = (a^2 - 7b^2 - 14ab) + i\sqrt{7}(a^2 - 7b^2 + 2ab)$$

Equating real and imaginary parts

$$U = a^2 - 7b^2 - 14ab$$

$$y = a^2 - 7b^2 + 2ab$$

We obtain the non-zero distinct integral solution of (1) as

$$x(a,b) = -4a^2 - 24ab + 28b^2$$

$$y(a,b) = a^2 - 7b^2 + 2ab$$

$$z(a,b) = a^2 + 7b^2$$

PROPERTIES:

1. $x(A,1) - z(A,1) - 21t_{4,a} + 24p_{ra} = 21$
2. $y(A,1) - z(A,1) + 2t_{4,a} - 2p_{ra} \equiv 0 \pmod 7$
3. $y(A,1) + z(A,1) - 2p_{ra} = 0$

PATTERN: 2

'8' can also written as

$$8 = \frac{(5+i\sqrt{7})(5-i\sqrt{7})}{2^2} \qquad (7)$$

Substituting (5) & (7) in (4), we get

$$(U+i\sqrt{7}y)(U-i\sqrt{7}y) = (a+i\sqrt{7}b)^2(a-i\sqrt{7}b)^2\frac{(5+i\sqrt{7})(5-i\sqrt{7})}{2^2}$$

Consider the positive factor

$$(U+i\sqrt{7}y) = (a+i\sqrt{7}b)^2\frac{(5+i\sqrt{7})}{2}$$

$$(U+i\sqrt{7}y) = \frac{(5+i\sqrt{7})}{2}(a^2 - 7b^2 + 2i\sqrt{7}ab)$$

$$(U+i\sqrt{7}y) = \frac{5a^2 - 14ab - 35b^2 + i\sqrt{7}(a^2 + 10ab - 7b^2)}{2}$$

Equating real and imaginary parts

$$U = \frac{5a^2 - 14ab - 35b^2}{2}$$

$$y = \frac{a^2 + 10ab - 7b^2}{2}$$

Assume a=2A, b=2B in the above equation and in view of (3), we obtain the non-zero distinct integral solution of (1) as

$$x(A,B) = -128AB$$
$$y(A,B) = 2A^2 + 20AB - 14B^2$$
$$z(A,B) = 4A^2 + 28B^2$$

PROPERTIES:

1. $x(A,1) + y(A,1) - 110t_{4,A} + 108p_{rA} \equiv 0 \pmod 7$
2. $x(A,1) + z(A,1) - 132t_{4,A} + 128p_{rA} = 28$
3. $x(A,1) - z(A,1) - 124t_{4,A} + 128p_{rA} + 28 = 0$

PATTERN: 3

(4) can also written as

$$U^2 + 7y^2 = 8z^2 * 1 \quad (8)$$

'1' can also be written as

$$1 = \frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{4^2} \quad (9)$$

Write '8' as

$$8 = (1 + i\sqrt{7})(1 - i\sqrt{7}) \quad (10)$$

Substituting (9), (10) in (8), we get

$$(U + i\sqrt{7}y)(U - i\sqrt{7}y) = \left[(a + i\sqrt{7}b)^2 (a - i\sqrt{7}b)^2 (1 + i\sqrt{7})(1 - i\sqrt{7})\right] * \left[\frac{(3 + i\sqrt{7})(3 - i\sqrt{7})}{4^2}\right]$$

Consider the positive factor

$$(U + i\sqrt{7}y) = \left[(a + i\sqrt{7}b)^2 (1 + i\sqrt{7})\right] * \left[\frac{(3 + i\sqrt{7})}{4}\right]$$

$$(U + i\sqrt{7}y) = \left[(a^2 - 7b^2 + i2ab\sqrt{7})(1 + i\sqrt{7})\right] * \left[\frac{(3 + i\sqrt{7})}{4}\right]$$

$$(U + i\sqrt{7}y) = \frac{(-4a^2 + 28b^2 - 56ab) + i\sqrt{7}(4a^2 - 28b^2 - 8ab)}{4}$$

Equating real and imaginary parts of the above equation, we get

$$U = -a^2 + 7b^2 - 14ab$$
$$y = a^2 - 7b^2 - 2ab$$

We obtain the non-zero distinct integral solution of (1) as

$$x(A,B) = -6a^2 + 42b^2 - 4ab$$
$$y(A,B) = a^2 - 7b^2 - 2ab$$
$$z(A,B) = a^2 + 7b^2$$

PROPERTIES:

1. $x(A,1) + y(A,1) - t_{4,a} + 6p_{ra} = 35$
2. $x(A,1) + z(A,1) + t_{4,a} + 4p_{ra} \equiv 0 \pmod 7$
3. $y(A,1) + z(A,1) - 4t_{4,a} + 2p_{ra} = 0$

PATTERN: 4

'1'can also be written as

$$1 = \frac{(1 + 3i\sqrt{7})(1 - 3i\sqrt{7})}{8^2} \quad (11)$$

Substituting (10) and (11), we

$$(U + i\sqrt{7}y)(U - i\sqrt{7}y) = \left[(a + i\sqrt{7}b)^2 (a - i\sqrt{7}b^2)(1 + i\sqrt{7})(1 - i\sqrt{7})\right] * \left[\frac{(1 + 3i\sqrt{7})(1 - 3i\sqrt{7})}{8^2}\right]$$

Consider the positive factor

$$(U + i\sqrt{7}y) = \left[(a + i\sqrt{7}b)^2 (1 + i\sqrt{7})\right] * \left[\frac{(1 + 3i\sqrt{7})}{8}\right]$$

$$(U + i\sqrt{7}y) = \left[(a^2 - 7b^2 + i2ab\sqrt{7})(1 + i\sqrt{7})\right] * \left[\frac{(1 + 3i\sqrt{7})}{8}\right]$$

$$(U + i\sqrt{7}y) = \frac{(-5a^2 + 35b^2 - 14ab) + i\sqrt{7}(a^2 - 7b^2 - 10ab)}{8}$$

Equating real and imaginary parts of the above equation, we get

$$U = \frac{-5a^2 + 35b^2 - 14ab}{2}$$
$$y = \frac{a^2 - 7b^2 - 10ab}{2}$$

Assume a=5A, b=5B in the above equations, we obtain the non-zero integral solution of (1) as

$$x(A,B) = -20A^2 + 140B^2 + 72AB$$
$$y(A,B) = 2A^2 - 14B^2 - 20AB$$
$$z(A,B) = 4A^2 + 28B^2$$

PROPERTIES:

1. $x(A,1) + y(A,1) + 70t_{4,A} - 52p_{rA} \equiv 0 \pmod 2$
2. $x(A,1) + z(A,1) + 88t_{4,A} - 72p_{rA} = 168$
3. $y(A,1) + z(A,1) - 26t_{4,A} + 20p_{rA} = 14$

PATTERN: 5

(4) Can also be written as

$$(U^2 - z^2) = 7(z^2 - y^2)$$
$$(U + z)(U - z) = 7(z + y)(z - y) \quad (12)$$

Case: 1

Equation (12) can be written as

$$\frac{U + z}{7(z + y)} = \frac{z - y}{U - z} = \frac{\alpha}{\beta} \quad (13)$$

Which is equivalent to the system of double equation as

$$\left.\begin{aligned} \beta U - 7\alpha y + (-\alpha + \beta)z &= 0 \\ -\alpha U - \beta y + (\alpha + \beta)z &= 0 \end{aligned}\right\} \quad (14)$$

Solving (14) by method cross multiplication, we get

$$\left.\begin{aligned} U &= -7\alpha^2 + \beta^2 - 14\alpha\beta \\ y &= \alpha^2 - \beta^2 - 2\alpha\beta \\ z &= -7\alpha^2 - \beta^2 \end{aligned}\right\} \quad (15)$$

Substituting (15) in (3) and we obtain the non-zero distinct integer solution of (1) as

$$x(\alpha,\beta) = -42\alpha^2 + 6\beta^2 - 4\alpha\beta$$
$$y(\alpha,\beta) = 7\alpha^2 - \beta^2 - 2\alpha\beta$$
$$z(\alpha,\beta) = -7\alpha^2 - \beta^2$$

PROPERTIES:

1. $x(\alpha,1) + y(\beta,1) + 29t_{4,\alpha} + 6p_{r\alpha} = 5$
2. $x(\alpha,1) + z(\alpha,1) + 45t_{4,\alpha} + 4p_{r\alpha} \equiv 0 \pmod 5$
3. $y(\alpha,1) + z(\alpha,1) - 2t_{4,\alpha} + 2p_{r\alpha} + 2 = 0$

Case: 2

Equation (12) can be written as

$$\frac{U + z}{z + y} = \frac{7(z - y)}{U - z} = \frac{\alpha}{\beta} \quad (16)$$

which is equivalent to the system of double equation as

$$\left.\begin{aligned} \beta U - \alpha y + (-\alpha + \beta)z &= 0 \\ -\alpha U - 7\beta y + (\alpha + 7\beta)z &= 0 \end{aligned}\right\} \quad (17)$$

Solving (17) by method cross multiplication, we get

$$\left.\begin{array}{l} U=-\alpha^2+7\beta^2-14\alpha\beta \\ y=\alpha^2-7\beta^2-2\alpha\beta \\ z=-\alpha^2-7\beta^2 \end{array}\right\} (18)$$

Substituting (18) in (3) and the non-zero distinct integral solution of (1) are given by

$$x(\alpha,\beta)=-6\alpha^2+42\beta^2-4\alpha\beta$$
$$y(\alpha,\beta)=\alpha^2-7\beta^2-2\alpha\beta$$
$$z(\alpha,\beta)=-\alpha^2-7\beta^2$$

PROPERTIES:

1. $x(\alpha,1)+z(\alpha,1)+3t_{4,\alpha}+4p_{r\alpha}=35$
2. $x(\alpha,1)+y(\alpha,1)-t_{4,\alpha}+6p_{r\alpha}\equiv 0(\bmod 7)$
3. $y(\alpha,1)+z(\alpha,1)-2t_{4,\alpha}+2p_{r\alpha}\equiv 0(\bmod 2)$

Case: 3

(12) can be written as

$$\frac{U+z}{z-y}=\frac{7(z+y)}{U-z}=\frac{\alpha}{\beta} \quad (19)$$

which is equivalent to the system of double equations as

$$\left.\begin{array}{l} \beta U+\alpha y+(-\alpha+\beta)z=0 \\ -\alpha U+7\beta y+(\alpha+7\beta)z=0 \end{array}\right\} \quad (20)$$

Solving (2.20) by method of cross multiplication, we get

$$\left.\begin{array}{l} U=\alpha^2-7\beta^2+14\alpha\beta \\ y=\alpha^2-7\beta^2-2\alpha\beta \\ z=\alpha^2+7\beta^2 \end{array}\right\} \quad (21)$$

Substituting (21) in (3), the non-zero distinct integral solution of (1) is given by

$$x(\alpha,\beta)=-4\alpha^2+28+24\alpha\beta$$
$$y(\alpha,\beta)=\alpha^2-7\beta^2-2\alpha\beta$$
$$z(\alpha,\beta)=\alpha^2+7\beta^2$$

PROPERTIES:

1. $x(\alpha,1)+z(\alpha,1)-21t_{4,\alpha}-24p_{r\alpha}\equiv 0(\bmod 5)$
2. $x(\alpha,1)-z(\alpha,1)+29t_{4,\alpha}-24p_{r\alpha}=21$
3. $y(\alpha,1)+z(\alpha,1)-4t_{4,\alpha}+2p_{r\alpha}=0$

Case: 4

(12) can be written as

$$\frac{U-z}{z-y}=\frac{7(z+y)}{U+z}=\frac{\alpha}{\beta} \quad (22)$$

Which is equivalent to the system of double equations as

$$\left.\begin{array}{l} \beta U+\alpha y-(\alpha+\beta)z=0 \\ -\alpha U+7\beta y+(-\alpha+7\beta)z=0 \end{array}\right\} \quad (23)$$

Solving (2.23) by method of cross multiplication, we get

$$\left.\begin{array}{l} U=-\alpha^2+7\beta^2+14\alpha\beta \\ y=\alpha^2-7\beta^2+2\alpha\beta \\ z=\alpha^2+7\beta^2 \end{array}\right\} \quad (24)$$

Substituting (24) in (3), the non-zero distinct integral solution of (1) is given by

$$x(\alpha,\beta)=-6\alpha^2+42\beta^2+4\alpha\beta$$
$$y(\alpha,\beta)=\alpha^2-7\beta^2+2\alpha\beta$$
$$z(\alpha,\beta)=\alpha^2+7\beta^2$$

PROPERTIES:

1. $x(\alpha,1)+y(\alpha,1)+11t_{4,\alpha}-6p_{r\alpha}\equiv 0(\bmod 5)$
2. $x(\alpha,1)+z(\alpha,1)+9t_{4,\alpha}-4p_{r\alpha}=49$
3. $y(\alpha,1)+z(\alpha,1)-2p_{r\alpha}=0$

CONCLUSION

In this Dissertation, we have made an attempt to obtain infinitely many non-zero distinct integer solutions to special ternary quadratic Diophantine equation $x^2+10xy+32y^2 = 8z^2$. In each case, a few interesting relations among the solution are presented. As the Diophantine equations are rich in variety, one may search for integer solutions to higher degree Diophantine equations with multiple variables along with suitable properties.

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